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Some Physical Properties of Neutrino-Gravitational Fields

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Abstract

A new solution of the Einstein-neutrino field equations is given. This solution is of Plebanski class $[4N]_3$ and describes a beam of neutrinos propagating along straight geodesics but possessing an inherent angular momentum density. Another previously known solution is also examined, and using some calculations given by Bonnor it is concluded that a uniform beam of neutrinos is gravitationally stable and that two such beams radiating in the same sense do not interact.

1. *Introduction*

Much work has been done recently on the two-component neutrino field in the general theory of relativity. However, the only known exact solutions to the Einstein-neutrino field equations appear to be included in those given by Griffiths & Newing (1970a, b). Another exact solution has been given by Golubyatnikov (1970) with an interesting interpretation. A particular case of one of these solutions had previously been given by Penney (1965). All of these known solutions describe neutrino pure radiation fields. That is, their energy momentum tensors belong to the Plebanski class $[4N]_2$ (see Plebanski, 1964). Another interesting feature of these solutions is that the metric tensor for some of them may also be interpreted as admitting null electromagnetic fields or photon radiation.

In this paper I will first obtain a new solution to the Einstein-neutrino field equations which corresponds to the Plebanski class $[4N]_3$ and I will give a geometrical interpretation of this solution. I will also quote one of the exact solutions cited above. For both of these solutions I will apply some work done by Bonnor to investigate the physical properties of these fields.

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2. *The Field Equations*

The gravitational equations may be given in suitable units† by

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -E_{\mu\nu} \tag{2.1}$$

where $E_{\mu\nu}$ is the energy momentum tensor of the neutrino field, given by

$$E_{\mu\nu} = \frac{i}{4}\{\sigma_{\mu A\dot{B}}(\xi^A \xi^{\dot{B}}{}_{;\nu} - \xi^{\dot{B}} \xi^A{}_{;\nu}) + \sigma_{\nu A\dot{B}}(\xi^A \xi^{\dot{B}}{}_{;\mu} - \xi^{\dot{B}} \xi^A{}_{;\mu})\} \tag{2.2}$$

ξ^A is the two-component neutrino spinor and must satisfy the neutrino equation

$$\sigma^{\alpha}{}_{A\dot{B}}\,\xi^A{}_{;\alpha} = 0 \qquad \text{or} \qquad \xi^A{}_{;A\dot{B}} = 0 \tag{2.3}$$

The covariant derivative of a spinor being given by

$$\xi_{A;\nu} = \xi_{A,\nu} - \Gamma_\nu{}^B{}_A\,\xi_B$$

where the spinor affine connection $\Gamma_\nu{}^B{}_A$ is given by

$$\Gamma_\nu{}^B{}_A = \tfrac{1}{2}\sigma_\alpha{}^{B\dot{C}}(\sigma^\alpha{}_{\dot{C}A,\nu} + \Gamma^\alpha{}_{\nu\beta}\,\sigma^\beta{}_{\dot{C}A})$$

As a consequence of equations (2.3) the trace of the energy momentum tensor (2.2) is zero and therefore the curvature scalar, R in (2.1), must also be zero.

The neutrino flux vector may be taken to be

$$l_\mu = \xi_A\,\sigma_\mu{}^{A\dot{B}}\,\xi_{\dot{B}}$$

It is now possible to define a null tetrad about l_μ. This is equivalent to taking ξ_A as a basis spinor, defining a second basis spinor η_A such that

$$\xi_A\,\eta^B - \eta_A\,\xi^B = \delta_A{}^B$$

and defining the remaining tetrad vectors in terms of these spinors by

$$m_\mu = \xi_A\,\sigma_\mu{}^{A\dot{B}}\,\eta_{\dot{B}}, \qquad \bar{m}_\mu = \eta_A\,\sigma_\mu{}^{A\dot{B}}\,\xi_{\dot{B}}, \qquad n_\mu = \eta_A\,\sigma_\mu{}^{A\dot{B}}\,\eta_{\dot{B}}$$

A tetrad defined in this way will be referred to as a 'neutrino tetrad'. In order to interpret the neutrino field geometrically it is useful to define the vector

$$\begin{aligned} H_\mu &= \xi^A\,\xi_{A;\mu} = m^\alpha\,l_{\alpha;\mu} \\ &= \tau l_\mu + \kappa n_\mu - \rho m_\mu - \sigma \bar{m}_\mu \end{aligned}$$

where τ, κ, ρ and σ are the spin coefficients associated with the neutrino tetrad (see Newman & Penrose, 1962). If in this expansion $\kappa = 0$, then l_μ is a tangent vector to a null geodesic congruence and, putting $\rho = \theta + i\omega$, θ, ω and $|\sigma|$ are proportional to the expansion, twist and shear of that congruence respectively.

† Greek suffices will take the values 0, 1, 2, 3, Latin suffices the values 1, 2, 3 and capital Latin suffices the values 1, 2. The signature of the metric is taken to be −2. Partial differentiation will be denoted by a comma and covariant differentiation by a semicolon.

3. *An Exact Solution*

As in a previous paper (Griffiths & Newing, 1970b), I will consider a metric tensor of the form

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & h_1 & h_2 & h_3 \\ 0 & h_2 & -1 & 0 \\ 0 & h_3 & 0 & -1 \end{pmatrix} \tag{3.1}$$

where h_i are independent of the coordinate x^0. The non-zero components of the Ricci tensor for this metric are

$$R_{11} = h_{2,21} + h_{3,31} - \tfrac{1}{2}(h_{1,22} + h_{1,33}) - \tfrac{1}{2}\gamma^2$$
$$R_{12} = R_{21} = \tfrac{1}{2}\gamma_{,3}$$
$$R_{13} = R_{31} = -\tfrac{1}{2}\gamma_{,2}$$

where $\gamma = h_{3,2} - h_{2,3}$. The Pauli matrices in Minkowski space-time will be denoted by $\sigma_{(\mu)}{}^{\dot{A}B}$ and I will use the particular representation

$$\sigma_{(0)}{}^{\dot{A}B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_{(1)}{}^{\dot{A}B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_{(2)}{}^{\dot{A}B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_{(3)}{}^{\dot{A}B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The generalised Pauli matrices $\sigma_\mu{}^{\dot{A}B}$ may now be given in terms of those of Minkowski space by

$$\sigma_0{}^{\dot{A}B} = \frac{1}{\sqrt{h_1}}(\sigma_{(0)}{}^{\dot{A}B} + \sigma_{(1)}{}^{\dot{A}B}) \qquad \sigma_1{}^{\dot{A}B} = \sqrt{(h_1)}\,\sigma_{(0)}{}^{\dot{A}B}$$
$$\sigma_2{}^{\dot{A}B} = \frac{h_2}{\sqrt{h_1}}(\sigma_{(0)}{}^{\dot{A}B} + \sigma_{(1)}{}^{\dot{A}B}) + \sigma_{(2)}{}^{\dot{A}B} \qquad \sigma_3{}^{\dot{A}B} = \frac{h_3}{\sqrt{h_1}}(\sigma_{(0)}{}^{\dot{A}B} + \sigma_{(1)}{}^{\dot{A}B}) + \sigma_{(3)}{}^{\dot{A}B}$$

and the components of the spinor affine connection are given by

$$\Gamma_{0A}{}^{B} = 0$$
$$\Gamma_{1A}{}^{B} = \frac{h_{1,1}}{2h_1}\sigma_{(0)A\dot{C}}\,\sigma_{(1)}{}^{\dot{C}B} + \frac{1}{\sqrt{h_1}}(\tfrac{1}{2}h_{1,2} - h_{2,1})(\sigma_{(0)A\dot{C}} + \sigma_{(1)A\dot{C}})\,\sigma_{(2)}{}^{\dot{C}B}$$
$$+ \frac{1}{\sqrt{h_1}}(\tfrac{1}{2}h_{1,3} - h_{3,1})(\sigma_{(0)A\dot{C}} + \sigma_{(1)A\dot{C}})\,\sigma_{(3)}{}^{\dot{C}B} + \tfrac{1}{2}\gamma\sigma_{(2)A\dot{C}}\,\sigma_{(3)}{}^{\dot{C}B}$$
$$\Gamma_{2A}{}^{B} = \frac{h_{1,2}}{2h_1}\sigma_{(0)A\dot{C}}\,\sigma_{(1)}{}^{\dot{C}B} + \frac{\gamma}{4\sqrt{h_1}}(\sigma_{(0)A\dot{C}} + \sigma_{(1)A\dot{C}})\,\sigma_{(2)}{}^{\dot{C}B}$$
$$\Gamma_{3A}{}^{B} = \frac{h_{1,3}}{2h_1}\sigma_{(0)A\dot{C}}\,\sigma_{(1)}{}^{\dot{C}B} - \frac{\gamma}{4\sqrt{h_1}}(\sigma_{(0)A\dot{C}} + \sigma_{(1)A\dot{C}})\,\sigma_{(3)}{}^{\dot{C}B}$$

For convenience I will re-label the coordinates by

$$x^0 = v, \qquad x^1 = u, \qquad x^2 = x, \qquad x^3 = y.$$

Now the form of this metric has been chosen for null radiation fields such that the coordinate v is a measure of the distance in the direction of propagation. Using this same approach I choose coordinates such that the neutrino flux vector l^μ only has a component in the v direction. This requires that the neutrino spinor must have the form

$$\xi_A = \begin{pmatrix} p \\ -p \end{pmatrix} \tag{3.2}$$

and the flux vector is then

$$l_\mu = \sqrt{(2h_1)}\, p\bar{p}\delta_\mu{}^1$$

In this notation the neutrino equations (2.3) reduce to

$$\frac{\partial p}{\partial v} = 0, \qquad \left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\log p^4 h_1 = 0$$

which have the immediate solution

$$p = h_1{}^{-1/4} \exp\left[L(u) + M(x - iy)\right] \tag{3.3}$$

where L and M are arbitrary complex functions.

The neutrino energy momentum tensor (2.2) reduces to

$$E_{\mu\nu} = \frac{\sqrt{h_1}}{2\sqrt{2}}\{\delta_\mu{}^1(ip\bar{p}_{,\nu} - i\bar{p}p_{,\nu} - \gamma p\bar{p}\delta_\nu{}^1) + (ip\bar{p}_{,\mu} - i\bar{p}p_{,\mu} - \gamma p\bar{p}\delta_\mu{}^1)\,\delta_\nu{}^1\}$$

Now substituting in the value of p above and putting $L(u) = l(u) + i\lambda(u)$ the non-zero components of this are

$$E_{11} = \frac{1}{\sqrt{2}} e^{2l} e^{M+\bar{M}}(2\lambda' - \gamma)$$

$$E_{12} = E_{21} = \frac{1}{2\sqrt{2}} e^{2l} e^{M+\bar{M}}\, i(\bar{M}' - M')$$

$$E_{13} = E_{31} = -\frac{1}{2\sqrt{2}} e^{2l} e^{M+\bar{M}}(\bar{M}' + M')$$

where a prime denotes the first derivative: for example $\lambda' = d\lambda/du$. The two field equations $R_{12} = -E_{12}$ and $R_{13} = -E_{13}$ are consistent since the form of M requires that $E_{12,2} = -E_{13,3}$. Now combining the gravitational field equations we obtain that, since the neutrino equations are satisfied, a metric may be interpreted as admitting a neutrino field if functions $l(u)$, $\lambda(u)$ and $M(x - iy)$ can be found such that

$$\frac{1}{\sqrt{2}} e^{2l} e^{M+\bar{M}}(2\lambda' - \gamma) = \tfrac{1}{2}(h_{1,22} + h_{1,33}) + \tfrac{1}{2}\gamma^2 - h_{2,21} - h_{3,31} \tag{3.4}$$

and

$$\frac{1}{\sqrt{2}} e^{2l} e^{M+\bar{M}} M' = -\tfrac{1}{2}(\gamma_{,2} + i\gamma_{,3}) \tag{3.5}$$

One exact solution of these equations can be obtained by putting

$$h_2 = -\chi(u)\, y\left(\frac{1}{2} - \frac{r}{3a}\right) \tag{3.6}$$

$$h_3 = \chi(u)\, x\left(\frac{1}{2} - \frac{r}{3a}\right) \tag{3.7}$$

where $r = (x^2 + y^2)^{1/2}$, a is a constant and $\chi(u)$ may be any given function of u. It is convenient to consider this as an interior solution for the cylindrical region $r < a$. It can be seen immediately that

$$\gamma = \chi(u)\left(1 - \frac{r}{a}\right)$$

and the field equation (3.5) becomes

$$\frac{1}{\sqrt{2}} e^{2l} e^{M+\bar{M}} M' = \chi \frac{1}{a}\left(\frac{x+iy}{2r}\right)$$

Now the equation

$$M' e^{M+\bar{M}} = \frac{x+iy}{2ar}$$

has the solution

$$M = \tfrac{1}{2} \log \frac{1}{a}(x - iy) \tag{3.8}$$

in which case $e^{M+\bar{M}} = r/a$. Thus we may take this as a solution with

$$e^{2l} = \sqrt{(2)}\,\chi \tag{3.9}$$

Now substituting these into (3.4) we obtain

$$2\chi \frac{r}{a}(2\lambda' - \gamma) = h_{1,22} + h_{1,33} + \gamma^2$$

which reduces to

$$h_{1,22} + h_{1,33} = \chi^2\left(-1 + \frac{4\lambda'}{\chi}\frac{r}{a} + \frac{r^2}{a^2}\right)$$

and this has a solution

$$h_1 = \chi^2 r^2\left(-\frac{1}{4} + \frac{4\lambda'}{9\chi}\frac{r}{a} + \frac{1}{16}\frac{r^2}{a^2}\right) \tag{3.10}$$

Thus a metric given by (3.1), (3.6), (3.7) and (3.10) may be interpreted as admitting a neutrino field whose spinor (3.3) may be obtained exactly, up to a constant phase factor, from (3.8), (3.9) and (3.10) in terms of the functions $\chi(u)$ and $\lambda(u)$ which are defined by the metric.

4. *Interpretation*

In order to consider the geometrical properties of this field, I will take ξ_A in the form (3.2) as a basis spinor and choose a second basis spinor to be $\eta_A = \frac{1}{2p}\begin{pmatrix}1\\1\end{pmatrix}$. Using these I obtain the neutrino tetrad vectors

$$l_\mu = \surd(2h_1)p\bar{p}\delta_\mu{}^1, \qquad m_\mu = \frac{1}{\surd 2}\frac{p}{\bar{p}}(i\delta_\mu{}^2 + \delta_\mu{}^3)$$

It can immediately be seen that

$$H_\mu = m^\alpha l_{\alpha;\mu} = 0$$

which demonstrates that the flux vector l_μ is tangent to a null geodesic congruence for which the expansion, twist and shear are all zero.

The energy momentum tensor can now be written as

$$E_{\mu\nu} = Al_\mu l_\nu + Bl_{(\mu}m_{\nu)} + \bar{B}l_{(\mu}\bar{m}_{\nu)}$$

where

$$A = \frac{1}{2\surd 2}e^{-2l}e^{-(M+\bar{M})}(2\lambda' - \gamma), \qquad B = \frac{1}{\surd 2}e^{-2i\lambda}e^{-(M-\bar{M})}M'$$

It can thus be seen that this field is of Plebanski class $[4N]_3$ and that it cannot be strictly interpreted as possessing positive energy density in the sense defined by Griffiths & Newing (1971) or Wainwright (1971). Neither does it obey the causality condition introduced by Audretsch (1971). However it does strictly describe a neutrino gravitational field since it is a solution of the only two necessary field equations (2.1) and (2.3).

It may now be noted that this solution is of the class which has been interpreted by Bonnor (1970) as describing a spinning null fluid enclosed in the cylindrical region $r < a$ and moving with the fundamental velocity. There is no restriction on the function $\chi(u)$ and when this is chosen to be a suitable smooth pulse function we obtain a model for a particle of null fluid which Bonnor calls a 'spinning nullicon'. However, a spinning null fluid is not necessarily to be interpreted as a neutrino field. This may be seen since, for a metric of this type, Bonnor's solution for a spinning null fluid does not in general satisfy the neutrino gravitational field equations (3.4) and (3.5).

Following Bonner's method, I have obtained the following exact global solution for a stream of neutrinos enclosed in the cylindrical region $r < a$.

$$r > a \qquad h_1 = \chi^2 a^2\left\{\left(\frac{4\lambda'}{3\chi} - \frac{1}{4}\right)\log\frac{r}{a} + \frac{4\lambda'}{9\chi} - \frac{3}{16}\right\}$$

$$h_2 = -\tfrac{1}{6}\chi y\frac{a^2}{r^2}, \qquad h_3 = \tfrac{1}{6}\chi x\frac{a^2}{r^2}$$

$$r<a \qquad h_1 = \chi^2 r^2\left(-\frac{1}{4}+\frac{4\lambda'}{9\chi}\frac{r}{a}+\frac{1}{16}\frac{r^2}{a^2}\right)$$

$$h_2 = -\chi y\left(\frac{1}{2}-\frac{r}{3a}\right), \qquad h_3 = \chi x\left(\frac{1}{2}-\frac{r}{3a}\right)$$

The relevant boundary conditions are satisfied since h_i, $\partial h_i/\partial x$ and $\partial h_i/\partial y$ are all continuous over the surface $r = a$. The exterior solution is a vacuum solution since $R_{\mu\nu} = 0$ for $r > a$.

I will now give a brief summary of Bonnor's interpretation of this field since he has shown that it possesses some interesting properties. The exterior solution may be interpreted as describing a plane-fronted gravitational wave. Since such waves can be generated by a null fluid without spin, the exterior solution is locally isometric to an exterior solution for a non-spinning null fluid. For the interior solution Bonnor has considered the transformation

$$\sqrt{(2)}\,u = t - z, \qquad \sqrt{(2)}\,v = t + z$$

which takes the metric into

$$ds^2 = dt^2 - dz^2 - dx^2 - dy^2 + \frac{h_1}{2}(dt-dz)^2 + \sqrt{(2)}\,(h_2\,dx + h_3\,dy)(dt-dz)$$

When the h_i are small this may be considered as a perturbation of Minkowski space-time. Now I will re-label the new coordinates by

$$\overline{x^0} = t, \qquad \overline{x^1} = z, \qquad \overline{x^2} = x, \qquad \overline{x^3} = y,$$

and the energy momentum tensor in this coordinate system can be given the usual classical interpretation by the linear approximation theory. In particular $\bar{E}_{02}$ and $\bar{E}_{03}$ can be interpreted as components of momentum density and since the integral of these over the cross-section is zero we understand that there is no total linear momentum in the plane of cross-section. However, the angular momentum about the direction of propagation per unit length is given by

$$\iint (y\bar{E}_{02} - x\bar{E}_{03})\,dx\,dy = \frac{\chi}{2\sqrt{2}}\iint \frac{r}{a}\,dx\,dy$$
$$= \frac{\pi a^2 \chi}{3\sqrt{2}}$$

Hence it follows that $(1/2\sqrt{2})\,\chi(r/a)$ is the corresponding density of angular momentum. Bonnor has pointed out that since γ vanishes on the boundary the angular momentum is equal to $(1/\sqrt{2})\iint \gamma\,dx\,dy$ and so $(1/\sqrt{2})\gamma = (1/\sqrt{2})\,\chi(1 - r/a)$ could also be the angular momentum density. However, the neutrino flux vector is now given by

$$l_\mu = \sqrt{(2)}\,e^{2l}\,e^{M+\bar{M}}\,\delta_\mu{}^1 = 2\chi\frac{r}{a}\delta_\mu{}^1$$

and since this is proportional to r/a the angular momentum density is more likely to be $(1/2\sqrt{2})\chi(r/a)$.

Using Bonnor's definition of energy density

$$\rho = \bar{E}_0{}^0 = \tfrac{1}{4}(h_{1,22} + h_{1,33} + \gamma^2 - h_2\gamma_{,3} + h_3\gamma_{,2})$$
$$= \tfrac{1}{4}\chi^2\left\{\left(\frac{4\lambda'}{\chi} - \frac{5}{2}\right)\frac{r}{a} + \frac{7}{3}\frac{r^2}{a^2}\right\}$$

the total energy per unit length becomes

$$e(u) = \iint \rho\, dx\, dy$$
$$= \frac{\pi a^2}{2}\chi^2\left(\frac{4\lambda'}{3\chi} - \frac{1}{4}\right)$$

These expressions indicate the following interesting properties. The functions λ' and χ are independent functions of u and so four separate cases can be distinguished:

(1) When $\lambda' \geqslant \frac{5}{8}\chi$ the energy density is positive right across the cylinder.
(2) When $\frac{5}{8}\chi > \lambda' > \frac{3}{16}\chi$ the total energy per unit length of the cylinder is positive, but the energy density in the central region of the cylinder is negative.
(3) When $\frac{3}{16}\chi > \lambda' > \frac{1}{24}\chi$ there will be a large region in the centre where the energy density is negative. This will be surrounded by a region of positive energy density, but the total energy across the cylinder will be negative.
(4) When $\lambda' \leqslant \frac{1}{24}\chi$ the energy density is negative right across the cylinder.

Since the functions χ and λ' are defined independently by the metric, all four of these situations are possible even in the same stream of neutrinos at different positions. And since they are both functions of u the energy will propagate with fundamental velocity. The expression for the energy density has been given for a particular observer. It is possible that when this observer sees a positive energy density, another observer might see a negative energy density. Thus even when $\lambda' \geqslant \frac{5}{8}\chi$ this is not a positive energy density in the sense considered by Griffiths & Newing (1971).

5. *Another Exact Solution*

Consider now the simple metric

$$ds^2 = 2\,dv\,du + h\,du^2 - dx^2 - dy^2 \tag{5.1}$$

where h is independent of v. This is a particular case of the metric considered above and so the field equations may be obtained immediately from there. Since $R_{12} = 0$ and $R_{13} = 0$ in this case, we must have $M = 0$ in (3.3) and the neutrino spinor becomes

$$\xi_A = \begin{pmatrix} 1 \\ -1 \end{pmatrix} h^{-1/4} \exp\,[l(u) + i\lambda(u)]$$

The only remaining neutrino gravitational equation is (3.4) which becomes

$$h_{,22} + h_{,33} = 2\sqrt{(2)}\, e^{2l} \lambda' \tag{5.2}$$

which is independent of x and y and so h can be written

$$h = a(u)\,x^2 + b(u)\,xy + c(u)\,y^2 + d(u)\,x + e(u)\,y + f(u)$$

where a, b, c, d, e and f are arbitrary functions of u subject to the single condition

$$a(u) + c(u) = \sqrt{(2)}\, e^{2l} \lambda'$$

This is a neutrino pure radiation field which describes a plane wave, the neutrino flux vector being

$$l_\mu = \sqrt{(2)}\, e^{2l}\, \delta_\mu{}^1$$

This result has been given previously by Griffiths & Newing (1970a). Since the metric is divergent for large values of x and y, this solution is only appropriate as an interior solution.

6. *Interpretation*

The physical properties of some particular cases of the above solution have been extensively discussed by Bonnor (1969). He has started with the metric (5.1) and discussed the properties of solutions to the gravitational equations

$$R_{\mu\nu} = -E_{\mu\nu} = -2\rho\delta_\mu{}^1\,\delta_\nu{}^1 \tag{6.1}$$

where ρ is interpreted as the energy density of the field. He has interpreted the solutions to this equation as describing the gravitational field of a beam of light, and later (1970) he refers to these solutions as describing beams of null fluid or 'nullicons'. However if ρ is a function of u only then (6.1) is equivalent to the neutrino gravitational equation (5.2), and all solutions may be interpreted as admitting neutrino pure radiation fields; the neutrino spinor being defined such that

$$\frac{1}{\sqrt{2}} e^{2l} \lambda' = \rho(u)$$

In particular all the exact solutions given by Bonnor (1969) may be interpreted as neutrino fields, and therefore the properties he deduces for beams of light are also properties of beams of neutrinos. Namely:

(1) The gravitational field of neutrinos is twice that of a material source of the same energy density.
(2) The gravitational field of pulses or beams of neutrinos consists of plane-fronted gravitational waves.
(3) Parallel beams (or pulses) of neutrinos shining in the same sense do not interact.
(4) A uniform beam of neutrinos is gravitationally stable.

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